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Cohomological operators and covariant quantum superalgebras

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Abstract

We obtain an interesting realization of the de Rham cohomological operators of differential geometry in terms of the noncommutative q-superoscillators for the supersymmetric quantum group $GL_{qp}(1|1)$. In particular, we show that a unique quantum superalgebra, obeyed by the bilinears of fermionic and bosonic noncommutative q-(super)oscillators of $GL_{qp}(1|1)$, is exactly identical to that obeyed by the de Rham cohomological operators. A set of discrete symmetry transformations for a set of $GL_{qp}(1|1)$ covariant quantum superalgebras turns out to be the analogue of the Hodge duality $*$ operation of differential geometry. A connection with an extended Becchi–Rouet–Stora–Tyutin (BRST) algebra obeyed by the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges, the conserved ghost charge and a conserved bosonic charge (which is equal to the anticommutator of (anti-)BRST and (anti-)co-BRST charges) is also established.

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1. Introduction

The subject of noncommutative geometry and corresponding noncommutative field theories has attracted a great deal of interest during the past few years. Such an upsurge of interest has been thriving because of its very clean and cogent appearance in the context of brane configurations related to the dynamics of string theories. In fact, the end points of the open strings, trapped on the D-branes, turn out to be noncommutative[1] in the presence of an antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) potential that constitutes the 2-form (i.e. $B = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) B_{\mu\nu}$)

[1] It will be noted that, in the context of string theories and D-branes, it is Snyder's idea of noncommutativity [1] that has become pertinent and popular.

background field for the whole system under consideration [2]. The noncommutative supersymmetric gauge theories [3, 4] are found to be the low energy effective field theories for the D-branes discussed above. The experimental tests (e.g., noncommutative Aharanov–Bohm effect, noncommutative synchrotron radiation, etc) for such kind of noncommutativity in the spacetime structure have been proposed [5, 6] where it has been argued that *only* the quantum mechanical effects are good enough to shed some light on its very existence. This is why, in the recent past, a whole range of quantum mechanical studies has been performed for the noncommutative (non-)relativistic systems and the ensuing results have gone into a systematic understanding of this subject from various physical and mathematical points of view (see, e.g., [7–10] and references therein).

The ideas behind the noncommutative (NC) spacetime and spacetime quantization are quite old ones (see, e.g., [1, 11, 12] for details). A few decades ago, it was conjectured that the deformation of groups (i.e., the subject of quantum groups) [13–17], based on the quasi-triangular Hopf algebras [18], together with the idea of noncommutative geometry might shed some light on the existence of a 'fundamental length' in the context of spacetime quantization. It was also hoped that this fundamental length will be responsible for getting rid of the infinities that plague the *local* quantum field theories (see, e.g., [12] for more details). In our present investigation, we address some of the interesting issues associated with the noncommutativity present in the subject of quantum groups (without going into any kind of discussion on Snyder's idea of noncommutativity). It is worthwhile, in the context of quantum groups, to recall that some interesting attempts have been made to construct the dynamics on an NC quantum phase space by exploiting the differential geometry and differential calculi developed on the NC quantum hyperplanes residing in the NC quantum cotangent manifolds (see, e.g., [19–22] and references therein). In particular, in [22], a consistent dynamics is constructed for the (non-)relativistic physical systems where a specific quantum group invariance and the ordinary (rotational) Lorentz invariance are respected together for any arbitrary ordering of the (space and) Lorentz spacetime indices. In a recent paper [23], the noncommutativity due to the quantum groups and the noncommutativity due to the presence of a magnetic field in the two-dimensional (2D) Landau problem are brought together in the construction of a consistent Hamiltonian and Lagrangian formulation where the symplectic structures, defined on the four-dimensional (4D) cotangent manifold, play a very important role. In this paper, it has been attempted to establish a connection between both kinds of noncommutativities (see, e.g., [23] for details). The q-deformed groups have also been treated as the gauge groups to develop the q-deformed Yang–Mills theories which reduce to the ordinary Yang–Mills gauge theories in the limit $q \to 1$ (see, e.g., [24, 25] and references therein for details). In these endeavours, the idea of quantum trace, quantum gauge orbits, quantum gauge transformations, etc have played notable roles [25, 26].

The purpose of our present paper is to establish, in a single theoretical setting, the inter-connections among (i) the de Rham cohomological operators of differential geometry, (ii) the $N = 2$ quantum mechanical superalgebra and (iii) the extended BRST algebra for some duality invariant gauge theories in the language of *noncommutative* q-superoscillators for the supersymmetric quantum group $GL_{qp}(1|1)$. We show that the bilinears of the noncommutative q-superoscillators of the supersymmetric quantum group $GL_{qp}(1|1)$ obey an algebra that is reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry. It is also demonstrated that the $GL_{qp}(1|1)$ covariant quantum superalgebras, obtained in our earlier work [26], are *unique* and they reduce to a *unique* superalgebra for the condition $pq = 1$. The latter remains covariant, as is quite obvious, under the co-action of the supersymmetric quantum group $GL_{q,q^{-1}}(1|1)$ and the bilinears of the q-superoscillators of this quantum group obey an algebra that is reminiscent of the

$N = 2$ supersymmetric quantum mechanics (SQM). At the SQM level too, an analogy with the de Rham cohomological operators is made, concentrating on the algebraic structure. The discrete symmetry transformations for all three covariant quantum superalgebras turn out to be the analogue of the Hodge duality $*$ operation of the differential geometry. This claim has been shown at the level of the conserved and nilpotent charges corresponding to the $N = 2$ supersymmetric quantum mechanical algebra (cf (6.10) and (6.11) below) as well as at the level of the conserved and nilpotent (co-)BRST charges (and corresponding nilpotent symmetry transformations) for the duality invariant gauge theories (cf (6.4), (6.6) and (6.7) below) and the corresponding extended BRST algebra[2]. The identifications of the supercharges as well as the (co-)BRST charges with the de Rham cohomological operators are in terms of the bilinears of the noncommutative q-superoscillators of $GL_{qp}(1|1)$.

Besides the motivations pointed out above, our present study is essential primarily on three counts. First, as is evident, the differential geometry and differential calculi play key roles in the discussion of a consistent dynamics in the framework of the Hamiltonian and/or Lagrangian formulation. Thus, it is an interesting endeavour to get some new noncommutative realization of the operators of the differential geometry which might play important roles in the description of the consistent noncommutative dynamics (see, e.g., section 7 for more discussions). Second, to the best of our knowledge, the noncommutative realization of the cohomological operators, Hodge duality $*$ operation, Hodge decomposition theorem, etc, has not been achieved so far in the language of the quantum groups. It is, therefore, a challenging problem to obtain such a realization. Finally, our present investigation *might* turn out to be useful in the description of the q-deformed gauge theories where the language of groups, differential geometry and differential forms is exploited extensively. Furthermore, such studies might complement (or provide an alternative to) the progress made in the realm of NC gauge theories based on Snyder's idea of noncommutativity.

The contents of our present investigation are organized as follows. In section 2, we present a convenient synopsis of some key concepts connected with the de Rham cohomological operators. For our present paper to be self-contained, in section 3, we recapitulate some preliminary results of our earlier work [26] in a somewhat *different* manner. We derive a couple of covariant superalgebras for $GL_{qp}(1|1)$ in section 4. Sections 5 and 6 are central to our present paper. We deal with the discrete symmetries for the covariant superalgebras in section 5. These are shown to correspond to the Hodge duality $*$ operation of differential geometry in section 6. Furthermore, in section 6, we also show the connection of some specific bilinears of the q-superoscillators of $GL_{qp}(1|1)$ and their $N = 2$ SQM algebra with the BRST operators and their extended algebra. We make some concluding remarks in section 7 and point out a few future directions that could be pursued later.

2. Preliminary: de Rham cohomological operators

On a compact D-dimensional manifold without a boundary, there exist three (i.e. $\mathrm{d}, \delta, \Delta$) cohomological operators in the realm of differential geometry which are found to be responsible for the study of the key and crucial properties associated with the differential forms defined on the manifold. The (co-)exterior derivatives $(\delta)\,\mathrm{d}$ are nilpotent of order 2 (i.e. $\mathrm{d}^2 = \delta^2 = 0$) which could be readily proved by exploiting the basic definitions ($\mathrm{d} = \mathrm{d}x^\mu \partial_\mu, \delta = \pm * \mathrm{d} *, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, etc) of these operators in the inner product of the differential forms defined on the D-dimensional compact manifold

[2] A concise discussion about the (co-)BRST symmetries, corresponding nilpotent (co-)BRST charges and their extended BRST algebra, etc, for the duality invariant gauge theories, has been given in section 6. Appropriate references on this topic are cited at the beginning of this section.

(i.e. $\mu = 0, 1, 2, \ldots, D-1$). Here $*$ is the Hodge duality operation defined on the manifold. The Laplacian operator $\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\,\mathrm{d}$ is defined in terms of the nilpotent (co-)exterior derivatives and is a self-adjoint and a positive semi-definite quantity for a given compact manifold. The algebra obeyed by the above operators can be succinctly expressed as

$$\begin{array}{lll} \mathrm{d}^2 = 0 & \delta^2 = 0 & \Delta = (\mathrm{d}+\delta)^2 = \{\mathrm{d}, \delta\} \\ [\Delta, \mathrm{d}] = 0 & [\Delta, \delta] = 0 & \{\mathrm{d}, \delta\} \neq 0. \end{array} \tag{2.1}$$

The above algebra shows that the Laplacian operator Δ is the Casimir operator for the whole algebra because it commutes with all the cohomological operators [27–29].

The de Rham cohomology groups characterize the topology of a given manifold in terms of the key properties associated with the differential forms. These properties are, in a subtle way, captured by the cohomological operators d, δ, Δ. In this context, it is pertinent to point out that some of the properties that owe their origin to the cohomological operators are as follows: (i) a differential form f_n of degree n is said to be closed ($\mathrm{d}f_n = 0$) and co-closed ($\delta f_n = 0$) if it is annihilated by d and δ, respectively; (ii) the same form is said to be exact ($f_n = \mathrm{d}e_{n-1}$) and co-exact ($f_n = \delta c_{n+1}$) if and only if the above closed ($\mathrm{d}f_n = 0$) and co-closed ($\delta f_n = 0$) conditions are satisfied trivially due to the nilpotency ($\mathrm{d}^2 = \delta^2 = 0$) of the (co-)exterior derivatives (δ) d; (iii) an n-form (h_n) is said to be a harmonic form if the Laplace equation $\Delta h_n = 0$ is satisfied which finally implies that the harmonic form h_n is closed ($\mathrm{d}h_n = 0$) and co-closed ($\delta h_n = 0$), simultaneously; (iv) the celebrated Hodge decomposition theorem, on a compact manifold without a boundary, can be defined in terms of the de Rham cohomological operators (d, δ, Δ) as (see, e.g., [27–29] for details)

$$f_n = h_n + \mathrm{d}e_{n-1} + \delta c_{n+1} \tag{2.2}$$

which states that any arbitrary n-form f_n (with $0 \leqslant n \leqslant D$; $n = 0, 1, 2 \ldots$) on a D-dimensional compact manifold can be uniquely written as the sum of a harmonic form h_n, an exact form $\mathrm{d}e_{n-1}$ and a co-exact form δc_{n+1}. A close look at (2.2) demonstrates that the degree of a form f_n is raised by 1 if the exterior derivative d acts on it (i.e. $\mathrm{d}f_n \sim g_{n+1}$). In contrast, the degree of a form f_n is lowered by 1 if it is acted upon by the co-exterior derivative δ (i.e. $\delta f_n \sim g_{n-1}$). The degree of a form f_n remains intact if it is acted upon by the Laplacian operator Δ (i.e. $\Delta f_n \sim g_n$).

Two closed ($\mathrm{d}f' = \mathrm{d}f = 0$) forms f and f' are said to belong to the same cohomology class with respect to the exterior derivative d if they differ by an exact form (i.e. $f' = f + \mathrm{d}g$, for an appropriate non-zero form g). Similarly, a co-cohomology can be defined w.r.t. δ where any arbitrary two co-closed ($\delta c' = \delta c = 0$) forms c' and c differ by a co-exact form (i.e. $c' = c + \delta m$, for an appropriate non-zero form m). To wrap up this section, we comment briefly on the $\pm$ signs present in the relationship ($\delta = \pm * \mathrm{d} *$) between the (co-)exterior derivatives (δ) d. By taking the inner product of the forms on the D-dimensional manifold, it can be shown that, for an even D, we obtain the relationship between δ and d with a minus sign (i.e. $\delta = - * \mathrm{d} *$). This conclusion is dictated by the fact that, in general, an inner product between n-forms on the D-dimensional manifold leads to the relationship between δ and d as (see, e.g., [27] for details)

$$\delta = (-1)^{(Dn+D+1)} * \mathrm{d} * . \tag{2.3}$$

Thus, for an even-dimensional manifold, there is always a minus sign on the rhs and for the odd-dimensional manifold, the above relation becomes $\delta = (-1)^n * \mathrm{d} *$ which shows that the $\pm$ signs on the rhs for the latter case depend on the degree of the forms that are involved in the specific inner product defined on the odd-dimensional manifold.

3. Quantum group $GL_{qp}(1|1)$ and q-superoscillators

In this section, we very briefly recapitulate, in a somewhat different manner, the bare essentials of our earlier work [26] which will be relevant for our further discussions. It can be seen that the following transformations,

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv (T) \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.1}$$

$$(\tilde{x} \quad \tilde{y}) \rightarrow (\tilde{x}' \quad \tilde{y}') = (\tilde{x} \quad \tilde{y}) \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \equiv (\tilde{x} \quad \tilde{y})(T) \tag{3.2}$$

for the even pair of variables $(x, \tilde{x})$ and odd $(y^2 = \tilde{y}^2 = 0)$ pair of variables $(y, \tilde{y})$, that define the super quantum hyperplane, with conditions

$$\begin{aligned} &xy = qyx \rightarrow x'y' = qy'x' \qquad && y^2 = 0 \rightarrow (y')^2 = 0 \\ &\tilde{x}\tilde{y} = p\tilde{y}\tilde{x} \rightarrow \tilde{x}'\tilde{y}' = p\tilde{y}'\tilde{x}' \qquad && \tilde{y}^2 = 0 \rightarrow (\tilde{y}')^2 = 0 \end{aligned} \tag{3.3}$$

lead to the braiding relationships among the rows and columns, constituted by the even elements (a, d) and odd $(\beta^2 = \gamma^2 = 0)$ elements (β, γ) of the 2×2 supersymmetric quantum matrix T defined in (3.1) and (3.2), as

$$\begin{aligned} &a\beta = p\beta a \qquad && a\gamma = q\gamma a \qquad && \beta\gamma = -(q/p)\gamma\beta \qquad && d\gamma = q\gamma d \\ &d\beta = p\beta d \qquad && \beta^2 = \gamma^2 = 0 \qquad && ad - da = -(p - q^{-1})\beta\gamma = (q - p^{-1})\gamma\beta. && \end{aligned} \tag{3.4}$$

In fact, this method is a simplified version of Manin's super quantum hyperplane approach to the construction of the general quantum supergroups [17]. For $q = p$, the above relations boil down to the braiding relations in the rows and columns for the supersymmetric quantum group $GL_q(1|1)$ with a single deformation parameter q. For the special case of $pq = 1$, which corresponds to the supersymmetric quantum group $GL_{q,q^{-1}}(1|1)$, we obtain the following relations among the elements of $GL_{q,q^{-1}}(1|1)$ (that emerge from (3.4)):

$$\begin{aligned} &a\beta = q^{-1}\beta a \qquad && a\gamma = q\gamma a \qquad && \beta\gamma = -(q^2)\gamma\beta \qquad && d\gamma = q\gamma d \\ &d\beta = q^{-1}\beta d \qquad && \beta^2 = 0 \qquad && \gamma^2 = 0 \qquad && ad = da. \end{aligned} \tag{3.5}$$

The above relations (3.4) and (3.5) will turn out to be quite useful for our later discussions. To study the $GL_{qp}(1|1)$ covariant relations among the q-superoscillators, we introduce a pair of noncommutative bosonic oscillators $(A, \tilde{A})$ and a pair of noncommutative fermionic (i.e. $B^2 = \tilde{B}^2 = 0$) oscillators $(B, \tilde{B})$. It is straightforward to check that the following $GL_{qp}(1|1)$ transformations of the super column matrix $(A, B)^T$ and the super row matrix $(\tilde{A}, \tilde{B})$ constructed by the superoscillators

$$\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \equiv (T) \begin{pmatrix} A \\ B \end{pmatrix} \tag{3.6}$$

$$\begin{aligned} (\tilde{A} \quad \tilde{B}) \rightarrow (\tilde{A}' \quad \tilde{B}') &= (\tilde{A} \quad \tilde{B}) \begin{pmatrix} a^{-1}(1 + \beta d^{-1}\gamma a^{-1}) & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1} & d^{-1}(1 - \beta a^{-1}\gamma d^{-1}) \end{pmatrix} \\ &\equiv (\tilde{A} \quad \tilde{B})(T)^{-1} \end{aligned} \tag{3.7}$$

leave the following algebraic relationships invariant,

$$AB = qBA \qquad \tilde{B}\tilde{A} = p\tilde{A}\tilde{B} \qquad B^2 = \tilde{B}^2 = 0 \tag{3.8}$$

if we exploit the q-commutation relations of (3.4). Consistent with (3.8), the other general covariant relations among the superoscillators are

$$
\begin{aligned}
&A\tilde{B} = \frac{(\lambda-\nu)}{q}\tilde{B}A \qquad B\tilde{A} = \frac{(\lambda-\nu)}{p}\tilde{A}B \\
&A\tilde{A} - \frac{(\lambda-\nu)}{pq}\tilde{A}A = B\tilde{B} + \frac{(\lambda-\nu)}{pq}\tilde{B}B
\end{aligned}
\tag{3.9}
$$

if we assume the validity of the following general relation between the bilinears constructed from the bosonic oscillators A, $\tilde{A}$ as well as the fermionic oscillators B, $\tilde{B}$,[3]

$$
A\tilde{A} - \lambda\tilde{A}A = 1 + \nu\tilde{B}B \tag{3.10}
$$

where λ and ν are some arbitrary non-zero commuting parameters which can be determined by exploiting the *associativity* of the trilinear combinations of the (super)oscillators. This associativity requirement is, in fact, equivalent to invoking the sanctity of the graded Yang–Baxter equations *vis-à-vis* the covariant algebraic relations (cf (3.8)–(3.10)). It will be noted that all the relations in (3.9) actually emerge from (3.10) when we exploit the basic transformations (3.6) and (3.7) on the (super)oscillators and use the relations in (3.4) and (3.8). As a side remark, we wish to state that the last algebraic superoscillator relation of (3.9) and our assumption (3.10) imply the following relationship between the bilinears of the (super)oscillators,

$$
B\tilde{B} = 1 + \left(\lambda - \frac{\lambda-\nu}{pq}\right)\tilde{A}A + \left(\nu - \frac{\lambda-\nu}{pq}\right)\tilde{B}B \tag{3.11}
$$

where the rhs contains terms with all the tilde oscillators arranged towards the left and all the non-tilde oscillators arranged towards the right. This relationship will turn out to be quite helpful in the next section where we shall invoke the associativity condition. Taking into account the explicit transformations in (3.6), (3.7) and relations in (3.4), (3.8), it can be checked that all the (super)oscillator relations from (3.8) to (3.11) are *covariant* under the co-action of supersymmetric quantum group $GL_{qp}(1|1)$.

4. Covariant quantum superalgebras for $GL_{qp}(1|1)$

In this section, we establish the fact that a set of a couple of covariant superalgebras, obtained in our earlier work [26], is a *unique* set of algebras for the quantum group $GL_{qp}(1|1)$. To this end, we compute the exact values of the parameters λ and ν in the above from the requirement that in the set of, for instance, a trilinear (super)oscillator $B\tilde{B}\tilde{A}$, we can bring all the tilde operators to the left in two different ways as listed below:

$$
\begin{aligned}
(B\tilde{B})\tilde{A} = {} & \left[1 + \lambda - \frac{\lambda-\nu}{pq}\right]\tilde{A} + \lambda\left[\lambda - \frac{\lambda-\nu}{pq}\right]\tilde{A}\tilde{A}A \\
& + \left[\nu\left(\lambda - \frac{\lambda-\nu}{pq}\right) + (\lambda-\nu)\left(\nu - \frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{B}B
\end{aligned}
\tag{4.1}
$$

$$
B(\tilde{B}\tilde{A}) = [(\lambda-\nu)]\tilde{A} + (\lambda-\nu)\left[\lambda - \frac{\lambda-\nu}{pq}\right]\tilde{A}\tilde{A}A + \left[(\lambda-\nu)\left(\nu - \frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{B}B. \tag{4.2}
$$

At this crucial juncture, a couple of remarks are in order. First, it will be noted that in (4.1) as well as (4.2), we have chosen a different set of (super)oscillators than that chosen in our earlier

[3] It should be emphasized that, in our earlier work [26], we have postulated the validity of a different kind of relationship (i.e. $B\tilde{B} + \nu\tilde{B}B = 1 + \lambda\tilde{A}A$) among the bilinears of the bosonic and fermionic (super)oscillators.

work[4] [26]. Second, the expressions for the (super)oscillators on the rhs of equations (4.1) and (4.2) are unique as far as all the covariant algebraic relations among the (super)oscillators from (3.8) to (3.11) are concerned. For the validity of the associativity condition, it is essential that the rhs of both the above equations should match with each other. Such an equality imposes the following two conditions on λ and ν:

$$\text{(i)}\quad \nu = 0 \quad \lambda = pq \qquad \text{(ii)}\quad \nu = \frac{(1-pq)}{pq} \quad \lambda = \frac{1}{pq}. \tag{4.3}$$

It should be re-emphasized that the above *associativity* requirement is equivalent to the validity of the graded Yang–Baxter equation in the context of supersymmetric quantum group $GL_{qp}(1|1)$. The covariant superalgebra for the bilinears corresponding to case (i) is

$$\begin{gathered} B\tilde{A} = q\tilde{A}B \qquad A\tilde{B} = p\tilde{B}A \qquad A\tilde{A} - pq\tilde{A}A = 1 \\ B\tilde{B} + \tilde{B}B = 1 + (pq-1)\tilde{A}A \end{gathered} \tag{4.4}$$

which are in addition to the invariant relations (3.8) for the bilinears. For case (ii), in addition to (3.8), the other bilinear q-superoscillator relations are

$$\begin{gathered} B\tilde{A} = p^{-1}\tilde{A}B \qquad A\tilde{B} = q^{-1}\tilde{B}A \qquad B\tilde{B} + \tilde{B}B = 1 \\ A\tilde{A} - \frac{1}{pq}\tilde{A}A = 1 + \frac{(1-pq)}{pq}\tilde{B}B. \end{gathered} \tag{4.5}$$

It should be emphasized, at this stage, that relations (3.8), (4.4) and (4.5) are same as those obtained in our earlier work [26] where a different set of superoscillators (in the trilinear form) was taken into consideration. For the case when $pq = 1$, we obtain a *unique* solution ($\nu = 0, \lambda = 1$) where the algebraic relations (3.8), (4.4) and (4.5) reduce to

$$\begin{array}{llll} B\tilde{A} = q\tilde{A}B & A\tilde{B} = q^{-1}\tilde{B}A & A\tilde{A} - \tilde{A}A = 1 & B^2 = 0 \\ B\tilde{B} + \tilde{B}B = 1 & AB = qBA & \tilde{B}\tilde{A} = q^{-1}\tilde{A}\tilde{B} & \tilde{B}^2 = 0. \end{array} \tag{4.6}$$

From the q-superoscillators $(A, \tilde{A}, B, \tilde{B})$, one can construct a four-dimensional 'adjoint representation' for the supersymmetric quantum group $GL_{qp}(1|1)$ in terms of the following four bilinears [26],

$$Y = \frac{A\tilde{A} + \mu B\tilde{B}}{1+\mu} \qquad H = A\tilde{A} - B\tilde{B} \qquad Q = A\tilde{B} \qquad \bar{Q} = B\tilde{A} \tag{4.7}$$

where $\mu \neq -1$ and the specific operator $H = A\tilde{A} - B\tilde{B}$ is invariant under the co-action of the supersymmetric quantum group $GL_{qp}(1|1)$. It is worthwhile emphasizing that the above operator H has been derived in [26] by exploiting the idea of supertrace for a 2×2 super quantum matrix constructed from the q-superoscillators $A, \tilde{A}, B, \tilde{B}$. The operators in (4.7) obey the following superalgebra which turns out to be reminiscent of the $N = 2$ supersymmetric quantum algebra:

$$\begin{gathered} [H, Q] = [H, \bar{Q}] = [H, Y] = 0 \qquad Q^2 = \bar{Q}^2 = 0 \\ \{Q, \bar{Q}\} = H \qquad [Q, Y] = +Q \qquad [\bar{Q}, Y] = -\bar{Q}. \end{gathered} \tag{4.8}$$

The above supersymmetric algebra is true for the case when $\nu = (pq)^{-1}(1-pq)$, $\lambda = (pq)^{-1}$ (i.e. case (ii) in equation (4.3)) as well as for the case when $pq = 1$. In the latter case, both

[4] For the proof of associativity, a trilinear set $A\tilde{A}\tilde{B}$ has been chosen in [26] for the purpose of re-ordering it in two different ways. More such kinds of trilinear combinations of (super)oscillators can be considered for the determination of ν and λ (see, e.g., the appendix for details). However, the covariant quantum superalgebras (i.e. (3.8), (4.4), (4.5)) remain the same. This demonstrates clearly the *uniqueness* of these basic superalgebras that are present in (3.8), (4.4), (4.5) as well as in their special case (4.6).

the conditions of (4.3) reduce to a single condition (i.e. $\nu = 0, \lambda = 1$). Such a kind of algebra for the case when $\nu = 0, \lambda = pq$ (i.e. case (i) of (4.3)) is as follows:

$$\begin{aligned}
&[H, Q] = 0 \qquad [H, \bar{Q}] = 0 \qquad [H, Y] = 0 \qquad Q^2 = \tfrac{1}{2}\{Q, Q\} = 0\\
&\{Q, \bar{Q}\} = [1 + (pq - 1)H]H \qquad \bar{Q}^2 = \tfrac{1}{2}\{\bar{Q}, \bar{Q}\} = 0\\
&[Q, Y] = +[1 + (pq - 1)H]Q \qquad [\bar{Q}, Y] = -[1 + (pq - 1)H]\bar{Q}.
\end{aligned} \tag{4.9}$$

Even though (4.9) looks a bit different from (4.8), it can be seen that the following redefinitions of Y, Q, $\bar{Q}$ in terms of $\hat{Y}$, P, $\bar{P}$

$$\hat{Y} = \frac{Y}{1 + (pq - 1)H} \qquad P = \frac{Q}{[1 + (pq - 1)H]^{(1/2)}} \qquad \bar{P} = \frac{\bar{Q}}{[1 + (pq - 1)H]^{(1/2)}} \tag{4.10}$$

lead to the $N = 2$ supersymmetric quantum mechanical superalgebra

$$\begin{aligned}
&[H, P] = [H, \bar{P}] = [H, \hat{Y}] = 0 \qquad P^2 = \bar{P}^2 = 0\\
&\{P, \bar{P}\} = H \qquad [P, \hat{Y}] = +P \qquad [\bar{P}, \hat{Y}] = -\bar{P}.
\end{aligned} \tag{4.11}$$

The identification in (4.10) is valid for derivation of (4.11) because H is the Casimir operator for (4.9) and it does commute with the original operators Q, $\bar{Q}$ and Y. It is crystal clear that the four operators in (4.7) do give a realization of $N = 2$ supersymmetric quantum mechanics in terms of the *noncommutative* q-superoscillators. Here the Hamiltonian H is invariant under $GL_{qp}(1|1)$ transformations (3.6) and (3.7), Q and $\bar{Q}$ are like nilpotent supercharges and Y is like a Witten index which encodes the fermion number for the supersymmetric quantum mechanical theory.

At this stage, we summarize the main results of our present section. First, the superoscillator algebraic relations (3.8) remain *invariant* under the co-action of the supersymmetric quantum group $GL_{qp}(1|1)$. Second, the requirement of associativity condition leads to only two $GL_{qp}(1|1)$ *covariant* algebraic relations (cf (4.4) and (4.5) in addition to (3.8)) for the specific values of λ and ν as given in (4.3). Third, the above two *covariant* relations reduce to a unique algebraic relation (4.6) which is found to be $GL_{q,q^{-1}}(1|1)$ covariant under the co-action of $GL_{qp}(1|1)$ for the deformation parameters satisfying $pq = 1$. Fourth, this unique superalgebra provides a unique realization of the de Rham cohomology operators of differential geometry as there is one-to-one correspondence between bilinears of the q-superoscillators and the cohomological operators (i.e. $Q \to \mathrm{d}$, $\bar{Q} \to \delta$, $H \to \Delta$) as can be seen in (4.8). The operator Y is the analogue of the Witten index which determines the *degree of the forms* in the language of fermion numbers[5] of the supersymmetric theory (see, e.g., [31] for details). Fifth, the analogue of the Hodge duality $*$ operation of differential geometry turns out to be a host of discrete symmetry transformations for the superalgebras (3.8), (4.4), (4.5) and (4.6) which are discussed in section 6 (see below).

5. Discrete symmetries for the covariant superalgebras

It is interesting to note that the superalgebra (4.6) for $pq = 1$ (i.e. $\nu = 0, \lambda = 1$) remains form invariant under the following discrete symmetry transformations:

$$A \to \pm \mathrm{i}\tilde{A} \qquad \tilde{A} \to \pm \mathrm{i}A \qquad B \to \pm\tilde{B} \qquad \tilde{B} \to \pm B. \tag{5.1}$$

[5] For the supersymmetric quantum mechanical theory, it can be checked that one can choose $Q = \sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)$, $\bar{Q} = \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$, $Y = \frac{1}{2}(1 - \sigma_3)$ in terms of the 2×2 Pauli matrices which do satisfy $[Q, Y] = +Q$, $[\bar{Q}, Y] = -\bar{Q}$, $Q^2 = \bar{Q}^2 = 0$ (see, e.g., [30] for further references and more details).

Note that there is no transformation on the deformation parameters p and q but there are transformations on the superoscillators $A, \tilde{A}, B, \tilde{B}$. Similar kinds of a couple of discrete symmetry transformations for $\nu = 0, \lambda = pq$ in the case of covariant superalgebra (4.4) (together with relations (3.8)) are

$$A \to \pm\mathrm{i}q\tilde{A} \quad \tilde{A} \to \pm\mathrm{i}pA \quad B \to \pm q\tilde{B} \quad \tilde{B} \to \pm pB \quad q \to p^{-1} \quad p \to q^{-1} \tag{5.2a}$$

$$A \to \pm\mathrm{i}p\tilde{A} \quad \tilde{A} \to \pm\mathrm{i}qA \quad B \to \pm p\tilde{B} \quad \tilde{B} \to \pm qB \quad q \to p^{-1} \quad p \to q^{-1}. \tag{5.2b}$$

The covariant superalgebra (4.5) (together with (3.8)) corresponding to $\nu = (pq)^{-1}(1 - pq)$, $\lambda = (pq)^{-1}$ is found to be endowed with the following discrete symmetry transformations on the superoscillators $(A, \tilde{A}, B, \tilde{B})$ and the deformation parameters p and q:

$$A \to \pm\mathrm{i}\tilde{A} \quad \tilde{A} \to \pm\mathrm{i}A \quad B \to \pm\tilde{B} \quad \tilde{B} \to \pm B \quad q \to p^{-1} \quad p \to q^{-1}. \tag{5.3}$$

It can be checked that, under the transformations (5.1) and (5.3), the four bilinear operators of (4.7) individually undergo the following change:

$$\begin{aligned} &H = A\tilde{A} - B\tilde{B} \to \tilde{H} = -\tilde{A}A - \tilde{B}B \\ &Q = A\tilde{B} \to \tilde{Q} = \pm\mathrm{i}\tilde{A}B \qquad \bar{Q} = B\tilde{A} \to \tilde{\bar{Q}} = \pm\mathrm{i}\tilde{B}A \\ &Y = \frac{A\tilde{A} + \mu B\tilde{B}}{1+\mu} \to \tilde{Y} = \frac{-\tilde{A}A + \mu\tilde{B}B}{1+\mu}. \end{aligned} \tag{5.4}$$

It is elementary to check that the $N = 2$ supersymmetric quantum algebra (4.8), for the bilinears in (4.7), remains *form* invariant under (5.4). This can be succinctly stated in mathematical form as follows:

$$\begin{aligned} &[\tilde{H}, \tilde{Q}] = [\tilde{H}, \tilde{\bar{Q}}] = [\tilde{H}, \tilde{Y}] = 0 \qquad \tilde{Q}^2 = (\tilde{\bar{Q}})^2 = 0 \\ &\{\tilde{Q}, \tilde{\bar{Q}}\} = \tilde{H} \qquad [\tilde{Q}, \tilde{Y}] = +\tilde{Q} \qquad [\tilde{\bar{Q}}, \tilde{Y}] = -\tilde{\bar{Q}}. \end{aligned} \tag{5.5}$$

Now let us concentrate on the discrete transformations in (5.2). It is straightforward to see that the four bilinears of (4.7) transform in the following manner under (5.2):

$$\begin{aligned} &H = A\tilde{A} - B\tilde{B} \to \tilde{H} = -(pq)(\tilde{A}A + \tilde{B}B) \\ &Q = A\tilde{B} \to \tilde{Q} = \pm\mathrm{i}(pq)(\tilde{A}B) \qquad \bar{Q} = B\tilde{A} \to \tilde{\bar{Q}} = \pm\mathrm{i}(pq)(\tilde{B}A) \\ &Y = \frac{A\tilde{A} + \mu B\tilde{B}}{1+\mu} \to \tilde{Y} = -(pq)\left[\frac{\tilde{A}A - \mu\tilde{B}B}{1+\mu}\right]. \end{aligned} \tag{5.6}$$

These transformed operators obey the following algebra:

$$\begin{aligned} &[\tilde{H}, \tilde{Q}] = 0 \qquad [\tilde{H}, \tilde{\bar{Q}}] = 0 \qquad [\tilde{H}, \tilde{Y}] = 0 \qquad (\tilde{Q})^2 = \frac{1}{2}\{\tilde{Q}, \tilde{Q}\} = 0 \\ &\{\tilde{Q}, \tilde{\bar{Q}}\} = \left[1 + \left(\frac{1}{pq} - 1\right)\tilde{H}\right]\tilde{H} \qquad (\tilde{\bar{Q}})^2 = \frac{1}{2}\{\tilde{\bar{Q}}, \tilde{\bar{Q}}\} = 0 \\ &[\tilde{Q}, \tilde{Y}] = +\left[1 + \left(\frac{1}{pq} - 1\right)\tilde{H}\right]\tilde{Q} \qquad [\tilde{\bar{Q}}, \tilde{Y}] = -\left[1 + \left(\frac{1}{pq} - 1\right)\tilde{H}\right]\tilde{\bar{Q}}. \end{aligned} \tag{5.7}$$

As far as the superalgebras in (4.8), (4.9), (5.5) and (5.7) are concerned, there are a few comments in order. First, it can be noted that the deformation parameters do not appear in the algebra (4.8) for the cases (i) $\nu = 0, \lambda = 1$ (i.e. the case when $pq = 1$) and (ii) $\nu = (pq)^{-1}(1 - pq), \lambda = (pq)^{-1}$. As a result, the corresponding algebra (5.5) for the transformed bilinears remains form invariant. Second, in the case of $\nu = 0, \lambda = pq$, the superalgebra (4.9) contains deformation parameters for the bilinears of (4.7). This is why the corresponding superalgebra (5.7) for the tilde operators contains a $(pq)^{-1}$ in place of pq occurring in (4.9). The latter is due to the fact that, in the discrete transformations (5.2), we have $p \to q^{-1}, q \to p^{-1}$. Third, it is very interesting to point out that, for the restriction $pq = 1$, the discrete symmetry transformations in (5.3) do converge trivially to (5.1). Fourth, let us concentrate on (5.2*a*) which leads to $A \to \mathrm{i}q\tilde{A}$, $\tilde{A} \to \mathrm{i}q^{-1}A$, $B \to q\tilde{B}$, $\tilde{B} \to q^{-1}B$ if we choose the upper signs. It can be readily checked that the algebraic relations (4.6) remain invariant under the above discrete transformations too. The same can be checked to be true for the other transformations in (5.2*a*) and (5.2*b*) as well. The key point to be noted is that these transformations (cf (5.2*a*) and (5.2*b*)) owe their origin to (5.1) (i.e. for $pq = 1$) when one plays with some constant factors (e.g., q and q^{-1}) that are plugged in the transformations (5.1). Fifth, the discrete symmetry transformations in (5.1)–(5.3) would turn out to be the analogue of Hodge duality $*$ operation of differential geometry as we shall see in the next section. In fact, we shall establish this analogy at the level of a duality between supercharges Q and $\bar{Q}$ *themselves* as well as at the level of symmetry transformations generated by Q and $\bar{Q}$ which are cast in the language of BRST and co-BRST symmetries, respectively.

6. Connection with the extended BRST algebra

In a recent set of papers (see, e.g., [32–39]), a connection between the de Rham cohomology operators $(\mathrm{d}, \delta, \Delta)$ and (anti-)BRST charges $Q_{(a)b}$, (anti-)co-BRST charges $Q_{(a)d}$, a ghost charge Q_g and a bosonic charge $W = \{Q_b, Q_d\} = \{Q_{ab}, Q_{ad}\}$ has been established for (i) the free Abelian 1-form gauge theory [31–33], (ii) the self-interacting 1-form non-Abelian gauge theory (where there is no interaction between the matter fields and gauge field) [34, 35], (iii) the interacting 1-form $U(1)$ gauge theory where there is an interaction between the 1-form Abelian gauge field and the matter (Dirac) fields [36, 37], and (iv) the free Abelian 2-form gauge theory [38, 39], etc, in the language of symmetry properties for the Lagrangian density of these theories. In all the above examples of the field theoretic models, the algebra satisfied by the local and conserved charges is found to be

$$\begin{aligned} &[W, Q_r] = 0 \qquad r = g, b, ab, d, ad \qquad Q_b^2 = Q_d^2 = Q_{ab}^2 = Q_{ad}^2 = 0 \\ &W = \{Q_b, Q_d\} = \{Q_{ab}, Q_{ad}\} \qquad \{Q_b, Q_{ad}\} = \{Q_d, Q_{ab}\} = 0 \qquad \{Q_b, Q_{ab}\} = 0 \\ &\mathrm{i}[Q_g, Q_{b(ad)}] = +Q_{b(ad)} \qquad \mathrm{i}[Q_g, Q_{d(ab)}] = -Q_{d(ab)} \qquad \{Q_d, Q_{ad}\} = 0. \end{aligned} \tag{6.1}$$

The above algebra is exactly like the algebra obeyed by the de Rham cohomological operators with a two-to-one mapping between the conserved charges and cohomological operators: $Q_{b(ad)} \to \mathrm{d}$, $Q_{d(ab)} \to \delta$, $W = \{Q_b, Q_d\} = \{Q_{ab}, Q_{ad}\} \to \Delta$. For all the above models, a set of discrete symmetry transformations has been shown to correspond to the Hodge duality $*$ operation of differential geometry. Furthermore, the analogue of the Hodge decomposition theorem (2.2) has been derived in the quantum Hilbert space of states where the *ghost number* plays the role of the *degree* of the differential forms. Thus, the above examples provide an interesting set of field theoretical models for the Hodge theory where all the cohomological operators, Hodge duality $*$ operation, Hodge decomposition theorem, etc, are expressed in terms of the local, covariant and continuous (as well as discrete) symmetry transformations and their corresponding generators (i.e. conserved charges).

Now we shall concentrate on the *unique* algebra (4.8) which is separately valid for (i) $\nu = (pq)^{-1}(1 - pq), \lambda = (pq)^{-1}$ and (ii) $\nu = 0, \lambda = 1$ (i.e. the case when $pq = 1$). In fact, as emphasized earlier, both the covariant quantum superalgebras (4.8) and (4.9) converge to (4.8) for the case $pq = 1$. In contrast to the 'two-to-one' mapping between local and conserved charges and the cohomological operators for the algebra (6.1), we shall see that the algebra (4.8) provides a 'one-to-one' mapping between the bilinears of (4.7) and the conserved charges of the BRST formalism. Such a suitable identification, for our purpose, is

$$\begin{aligned} &Q_b = A\tilde{B} \equiv Q \qquad Q_d = B\tilde{A} \equiv \bar{Q} \qquad Q_b^2 = Q_d^2 = 0 \\ &W = (A\tilde{A} - B\tilde{B}) \equiv H \qquad -\mathrm{i}Q_g = \frac{A\tilde{A} + \mu B\tilde{B}}{1+\mu} \equiv Y \end{aligned} \tag{6.2}$$

and, in addition, the discrete symmetry transformations (5.1)–(5.3) provide the realization of the Hodge duality $*$ operation of differential geometry. We distinguish our realization of duality (from the usual differential geometry Hodge $*$ duality) by denoting it by a separate and different $\star$ operation. To corroborate the above identifications beyond merely an *algebraic equivalence*, we note that the conserved charges (i.e. $\dot{Q}_b = [Q_b, H] = 0$, $\dot{Q}_d = [Q_d, H] = 0$) generate the symmetry transformations s_b and s_d for the Hamiltonian H. These transformations are encoded in the nilpotent $\left(s_b^2 = s_d^2 = 0\right)$ operators s_b and s_d. The explicit form of the transformations generated by the conserved charges $Q_b = A\tilde{B}$ and $Q_d = B\tilde{A}$ for the *noncommutative* superoscillators $A, \tilde{A}, B, \tilde{B}$, for the given superalgebra (4.6) (i.e. $\nu = 0, \lambda = 1$), are

$$\begin{aligned} &s_b\tilde{A} = [\tilde{A}, Q_b] = -q^{-1}\tilde{B} + (1-q^{-1})\tilde{A}A\tilde{B} \\ &s_d\tilde{A} = [\tilde{A}, Q_d] = (1-q)\tilde{A}B\tilde{A} \qquad s_bA = [A, Q_b] = (q^{-1}-1)A\tilde{B}A \\ &s_dA = [A, Q_d] = qB + (q-1)B\tilde{A}A \\ &s_b\tilde{B} = \{\tilde{B}, Q_b\} = 0 \qquad s_d\tilde{B} = \{\tilde{B}, Q_d\} = q\tilde{A} + (1-q)\tilde{B}B\tilde{A} \\ &s_dB = \{B, Q_d\} = 0 \qquad s_bB = \{B, Q_b\} = A + (q^{-1}-1)B\tilde{B}A. \end{aligned} \tag{6.3}$$

It is very interesting to point out that the above transformations are connected to each other by a general formula for the generic noncommutative q-(super)oscillator Φ as

$$\tilde{s}_d\Phi = \pm \star \tilde{s}_b \star \Phi \tag{6.4}$$

where the + sign on the rhs is for $\Phi = B, \tilde{B}$ and the $-$ sign on the rhs is for $\Phi = A, \tilde{A}$ for *all* cases of superalgebras (3.8), (4.4)–(4.6). The above signs are dictated by the general requirement of a duality invariant theory (see, e.g., [40] for details). We would like to lay stress on the fact that the relationship in (6.4) is the analogue of such a type of relation that exists in the differential geometry as given by equation (2.3). It will be noted that, under all the above discrete transformations (i.e. (5.1)–(5.3)) corresponding to the $\star$ operation, the result of two successive $\star$ operations on the noncommutative q-superoscillators $(A, \tilde{A}, B, \tilde{B})$ of $GL_{qp}(1|1)$ is

$$\star(\star A) = -A \qquad \star(\star\tilde{A}) = -\tilde{A} \qquad \star(\star B) = +B \qquad \star(\star\tilde{B}) = +\tilde{B} \tag{6.5}$$

which decides the signatures present in (6.4). This observation should be contrasted with the $(\pm)$ signs present in the relation $\delta = \pm * \mathrm{d} *$ between (co-)exterior derivatives $(\delta)\,\mathrm{d}$ of the differential geometry where these signs are dictated by the dimensionality of the manifold on which these operators are defined. The expressions for $\tilde{s}_b$ and $\tilde{s}_d$ present in (6.4) are different for various kinds of superalgebras. In fact, relation (6.4) is valid for *all* the covariant superalgebras. For instance, in the case of $\nu = 0, \lambda = 1$ which corresponds to $pq = 1$, we have

$$\tilde{s}_d = (-\mathrm{i}q)^{(-1/2)}s_d \qquad \tilde{s}_b = (-\mathrm{i}q)^{(+1/2)}s_b. \tag{6.6}$$

It will be noted that the deformation parameters do not transform (cf (5.1)) in the above $\star$ operation corresponding to the covariant quantum superalgebra (4.8). For the cases $\nu = 0, \lambda = pq$ and $\nu = (pq)^{-1}(1-pq), \lambda = (pq)^{-1}$, we have

$$\tilde{s}_d = (+\mathrm{i}p)^{(+1/2)}s_d \qquad \tilde{s}_b = (-\mathrm{i}q)^{(+1/2)}s_b. \tag{6.7}$$

Note that, in the above transformations (cf (5.2) and (5.3)) corresponding to $\star$, the deformation parameters do transform and a close look at (6.6) and (6.7) demonstrates that in the limit $p = q^{-1}$, we get back (6.6) from (6.7).

The sanctity and correctness of the relationship (6.4) can be checked by computing the transformations generated by Q_b and Q_d. These transformations for the basic superoscillators $A, \tilde{A}, B, \tilde{B}$, for the algebra (3.8) and (4.4) (i.e. for the case $\nu = 0, \lambda = pq$), analogous to (6.3), are

$$\begin{aligned}
&s_b\tilde{A} = [\tilde{A}, Q_b] = (1 - p^2q)\tilde{A}A\tilde{B} - p\tilde{B} \\
&s_d\tilde{A} = [\tilde{A}, Q_d] = (1 - q)\tilde{A}B\tilde{A} \\
&s_bA = [A, Q_b] = (p - 1)A\tilde{B}A \qquad s_dA = [A, Q_d] = qB + (pq^2 - 1)B\tilde{A}A \\
&s_b\tilde{B} = \{\tilde{B}, Q_b\} = 0 \qquad s_dB = \{B, Q_d\} = 0 \\
&s_bB = \{B, Q_b\} = A + (1 - q)BA\tilde{B} + (pq - 1)A\tilde{A}A \\
&s_d\tilde{B} = \{\tilde{B}, Q_d\} = p^{-1}\tilde{A} + (1 - p^{-1})\tilde{B}B\tilde{A} + p^{-1}(pq - 1)\tilde{A}A\tilde{A}.
\end{aligned} \tag{6.8}$$

The analogue of transformations (6.3) and (6.8) for the case of superalgebra (3.8) and (4.5) that corresponds to $\nu = (pq)^{-1}(1 - pq), \lambda = (pq)^{-1}$, is

$$\begin{aligned}
&s_b\tilde{A} = [\tilde{A}, Qb] = (1 - q^{-1})\tilde{A}A\tilde{B} - q^{-1}\tilde{B} \\
&s_d\tilde{A} = [\tilde{A}, Q_d] = (1 - p^{-1})\tilde{A}B\tilde{A} \\
&s_bA = [A, Q_b] = (q^{-1} - 1)A\tilde{B}A \qquad s_dA = [A, Q_d] = p^{-1}B + p^{-1}(1 - p)B\tilde{A}A \\
&s_b\tilde{B} = \{\tilde{B}, Q_b\} = 0 \qquad s_d\tilde{B} = \{\tilde{B}, Q_d\} = p^{-1}\tilde{A} + (1 - p^{-1})\tilde{B}B\tilde{A} \\
&s_dB = \{B, Q_d\} = 0 \qquad s_bB = \{B, Q_b\} = A + (1 - q)BA\tilde{B}.
\end{aligned} \tag{6.9}$$

With the help of (6.5)–(6.7), it can be checked that the relationship (6.4) is satisfied for all the transformations (6.3), (6.8) and (6.9). It is straightforward, in view of the relationship $\delta = \pm * \mathrm{d} *$ between (co-)exterior derivatives of differential geometry, to claim that the nilpotent transformations $\tilde{s}_d$ and $\tilde{s}_b$ are dual to each other. In other words, the discrete symmetry transformations (5.1)–(5.3) for the covariant superalgebras (3.8), (4.4)–(4.6) corresponding to the $\star$ operation (for the transformations connected with the q-superoscillators as well as the deformation parameters) are the analogue of the Hodge duality $*$ operation of the differential geometry.

This relationship can also be established at the level of the conserved (i.e. $[H, Q] = [H, \bar{Q}] = 0$) supercharges Q and $\bar{Q}$ of the identification (4.7) that obey the algebra (4.8). For instance, in the case of the unique superalgebra for $pq = 1$, we have the following relationship between Q and $\bar{Q}$ through S and $\bar{S}$,

$$\bar{S}\Phi = \pm \star S \star \Phi \qquad \bar{S} = (-\mathrm{i}q)^{-1/2}\bar{Q} \qquad S = (-\mathrm{i}q)^{1/2}Q \tag{6.10}$$

where the $+$ sign on the rhs is for $\Phi = B, \tilde{B}$ and the $-$ sign on the rhs is for $\Phi = A, \tilde{A}$. Similar relations are valid for the cases of the covariant quantum superalgebras when $pq \neq 1$ (i.e. (3.8) together with (4.4) and (4.5)). However, in those cases, the S and $\bar{S}$ are defined as

$$\bar{S} = (+\mathrm{i}p)^{+1/2}\bar{Q} \qquad S = (-\mathrm{i}q)^{+1/2}Q. \tag{6.11}$$

It will be noted that the conservation of supercharges $Q, \bar{Q}$ can be recast in the language of the BRST-type transformations s_b and s_d which turn out to be the symmetry transformations for the Hamiltonian H as given below:

$$s_bH = [H, Q_b] = 0 \qquad s_dH = [H, Q_d] = 0 \qquad \{s_b, s_d\}H = [H, \{Q_b, Q_d\}] = 0. \tag{6.12}$$

This can also be re-expressed in the language of the de Rham cohomological operators because $H \to \Delta$ is the Casimir operator for the whole algebra as $[H, Q] = [H, \bar{Q}] = 0 \to [H, Q_b] = [H, Q_d] = 0 \to [\Delta, \mathrm{d}] = [\Delta, \delta] = 0$. Hence in a single theoretical setting, we have obtained a neat relationship among the BRST formalism, de Rham cohomological operators and the covariant quantum superalgebras that are constructed by the bilinears of the noncommutative q-superoscillators for the supersymmetric quantum group $GL_{qp}(1|1)$.

To wrap up this section, we comment on (i) the transformations generated by the operators Y and H of the identification in (6.2), and (ii) the analogy between the variation of the degree of a form due to the operation of the cohomological operators in differential geometry and the changes of the ghost number of a state in the quantum Hilbert space (QHS) due to the application of the conserved charges in the framework of the BRST formalism. Let us first concentrate on (i). The corresponding transformations can be computed for all the algebraic relations (3.8), (4.4), (4.5) and (4.6). For the sake of simplicity, however, let us focus only on the simple case of (4.6) (i.e. the case for $pq = 1$). For this algebra, Y and H generate transformations that are encoded in the following commutators:

$$\begin{array}{llll} [\tilde{A}, Y] = -(1/1+\mu)\tilde{A} & [A, Y] = +(1/1+\mu)A & & \\ [\tilde{B}, Y] = -(\mu/1+\mu)\tilde{B} & [B, Y] = +(\mu/1+\mu)B & & \\ [\tilde{A}, H] = -\tilde{A} & [A, H] = +A & [\tilde{B}, H] = -\tilde{B} & [B, H] = +B. \end{array} \tag{6.13}$$

On face value, both the above transformations look the same modulo some constant factors. However, a close look at the identification (6.2) clarifies that there is a clear-cut distinction between the two because of the presence of an i factor in the expression for $Y = -\mathrm{i}Q_g$. In fact, between the two transformations, one corresponds to a scale transformation and the other corresponds to the gauge transformation. This is consistent with the ghost transformations (generated by the conserved ghost charge) and the bosonic transformations (generated by a bosonic charge that turns out to be the analogue of the Casimir operator) for a duality invariant gauge theory described in the framework of the BRST formalism (see, e.g., [32–39] for details). Now let us concentrate on (ii). The conserved charges Q_b, Q_d and H (which have been realized in terms of the noncommutative q-superoscillators) can be elevated to the operators in the QHS. Any arbitrary state $|\Psi\rangle_n$ with ghost number n (i.e. $\mathrm{i}Q_g|\Psi\rangle_n = n|\Psi\rangle_n$) can be decomposed into a unique sum (in analogy with (2.2)) as

$$|\Psi\rangle_n = |\omega\rangle_n + Q_b|\chi\rangle_{(n-1)} + Q_d|\theta\rangle_{(n+1)} \tag{6.14}$$

where $|\omega\rangle_n$ is the harmonic state (i.e. $Q_b|\omega\rangle_n = Q_d|\omega\rangle_n = 0$) and the nilpotent operators $Q_{(b)d}$ raise and lower the ghost number of states $|\chi\rangle_{(n-1)}$ and $|\theta\rangle_{(n+1)}$ by 1, respectively. In more explicit and lucid language, it can be seen that for the above state $|\Psi\rangle_n$ with ghost number n, we have

$$\begin{aligned} \mathrm{i}Q_g Q_b|\Psi\rangle_n &= (n+1)Q_b|\Psi\rangle_n \\ \mathrm{i}Q_g Q_d|\Psi\rangle_n &= (n-1)Q_d|\Psi\rangle_n \\ \mathrm{i}Q_g H|\Psi\rangle_n &= (n)H|\Psi\rangle_n \end{aligned} \tag{6.15}$$

which shows that the ghost numbers for the states $Q_b|\Psi\rangle_n$, $Q_d|\Psi\rangle_n$ and $H|\Psi\rangle_n$ (generated by the conserved charges Q_b, Q_d and H) are $(n+1)$, $(n-1)$ and n, respectively. This also establishes the correctness of the identification (6.2) of the bilinears of the q-superoscillators with the conserved charges of the BRST formalism that, in turn, are connected with the de Rham cohomological operators of differential geometry.

7. Conclusions

The central result of our present paper is to provide a realization of the de Rham cohomological operators of differential geometry, Hodge duality $*$ operation, Hodge decomposition theorem, etc, in the language of (i) the noncommutative q-superoscillators, (ii) the covariant quantum superalgebras of the bilinears in q-superoscillators, and (iii) the discrete symmetry transformations on the q-superoscillators, etc, for a doubly deformed supersymmetric quantum group $GL_{qp}(1|1)$. An interesting observation in our present investigation is the fact that a *unique* covariant supersymmetric quantum algebra emerges from a couple of consistent $GL_{qp}(1|1)$ covariant superalgebras for the condition $pq = 1$ on the deformation parameters. Furthermore, the bilinears constructed from the noncommutative q-superoscillators provide (i) a realization of the $N = 2$ supersymmetric quantum mechanical algebra, and (ii) a realization of an extended BRST algebra where there is one-to-one mapping between a set of conserved charges $(Q_b, Q_d, W, -\mathrm{i}Q_g)$ of the BRST formalism and the conserved supercharges $(Q, \bar{Q})$, the Hamiltonian (H) and the Witten index Y (that constitute a set $(Q, \bar{Q}, H, Y)$) of the $N = 2$ supersymmetric quantum mechanics, respectively. These charges, in turn, are connected with the de Rham cohomological operators of the differential geometry. Thus, our present investigation sheds light on the inter-connections among the de Rham cohomological operators of differential geometry, an extended BRST algebra (constituted by several conserved charges) for a class of duality invariant gauge theories and the $N = 2$ supersymmetric quantum mechanical algebra. All the above conserved charges and other operators are expressed in the language of noncommutative q-superoscillators of a doubly deformed supersymmetric quantum group $GL_{qp}(1|1)$ in their various guises.

It is worth emphasizing that the Hodge duality $*$ operation of the differential geometry appears in our discussion as a set of discrete symmetry transformations under which a set of $GL_{qp}(1|1)$ covariant superalgebras remains form invariant. This analogy and identification have been established at two different and distinct levels of our discussion. First, it turns out that the nilpotent (i.e. $Q^2 = \bar{Q}^2 = 0$) and conserved ($\dot{Q} = [Q, H] = 0$, $\dot{\bar{Q}} = [\bar{Q}, H] = 0$) supercharges Q and $\bar{Q}$ (modulo some constant factors) are connected (cf (6.10)) with each other in exactly the same manner as the (co-)exterior derivatives $(\delta)\, \mathrm{d}$ are related (i.e. $\delta = \pm * \mathrm{d} *$) to each other. Second, it is evident that the BRST-type transformations s_d and s_b, generated by $Q_d \equiv \bar{Q}$ and $Q_b \equiv Q$, are related (cf (6.4)), modulo some constant factors, in exactly the same way as the relationship (i.e. $\delta = \pm * \mathrm{d} *$) between (co-)exterior derivatives $(\delta)\, \mathrm{d}$ of differential geometry defined on a manifold without a boundary. At both levels of identifications, the $\star$ operation turns out to be equivalent to a set of discrete symmetry transformations (5.1)–(5.3), under which a set of covariant quantum superalgebras (cf (3.8), (4.4)–(4.6)) remains form invariant. The insight into such an identification comes basically from our experience with the duality invariant gauge theories (that present a set of tractable field theoretical models for the Hodge theory [32–39]) where the discrete symmetry transformations, for the (co-)BRST invariant Lagrangian densities, turn out to be the analogue of the Hodge duality $*$ operation of differential geometry.

It is very much essential for our present algebraic discussions to, ultimately, percolate down to the level of physical applications to some interesting dynamical systems. In this context, it is interesting to pinpoint that, in the language of differential geometry developed on the superquantum hyperplane (see, e.g., [41–43] and references therein for details), the noncommutative q-(super)oscillators can be identified with the Grassmannian as well as ordinary coordinates and the corresponding derivatives. For instance, the set of bosonic oscillators $(A, \tilde{A})$ can be identified with an ordinary coordinate x and the corresponding derivative $\partial/\partial x$, respectively. Similarly, the set of fermionic oscillators $(B, \tilde{B})$ can be identified with the Grassmannian coordinate θ and its corresponding derivative $(\partial/\partial\theta)$, respectively, where $\theta^2 = 0$ and $(\partial/\partial\theta)^2 = 0$. These identifications, in turn, allow us to get a differential calculus on the superquantum hyperplane from the covariant quantum algebra (3.8), (4.4)–(4.6) obeyed by the noncommutative q-(super)oscillators. The ensuing $GL_{qp}(1|1)$ covariant calculus will enable us to discuss physical systems on the superquantum hyperplane with deformation parameters p and q.

It is interesting to point out that a consistent formulation of the dynamics on a noncommutative phase space has been developed where the ordinary Lorentz (rotational) invariance and the noncommutative quantum group invariance are maintained *together* for the quantum group $GL_{qp}(2)$ with deformation parameters obeying $pq = 1$ [22, 23]. Some of these ingredients have also been exploited in the context of discussion of the Landau problem in two dimensions where Snyder's idea of noncommutativity (reflected due to the presence of a perpendicular constant magnetic field for a 2D electron system) and the noncommutativity due to quantum groups $GL_{qp}(2)$ with $pq = 1$ are present *together* [23]. The algebraic relations in the present paper and corresponding differential calculus might turn out to be useful in the discussion of a spinning relativistic particle on a deformed superhyperplane. In fact, this system has been discussed earlier [20] where only the on-shell conditions (i.e. the equations of motion) have been exploited to obtain the NC relations[6] among the phase variables which play a crucial role in the description of the dynamics on the noncommutative (q-deformed) phase space. However, a systematic and consistent differential calculus has not been developed in [20] on the super quantum hyperplane for such a discussion. We very strongly believe that our present work will bolster up the derivation of such a calculus on the super hyperplane. Similarly, some supersymmetric field theoretic models can also be discussed in a systematic manner by exploiting the differential calculus derived from the q-superoscillator algebra of our present paper. These are the key issues that are under investigation and our results will be reported elsewhere.

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Appendix

To establish the uniqueness of the algebras in (3.8), (4.4)–(4.6), we show that, given a trilinear combination of the (super)oscillators, we can arrange all the tilde oscillators to the left in two different ways due to the requirement of the associativity condition. In the following, the *pair* of (super)oscillators that are exchanged *first*, due to the covariant algebras given in section 3, are kept within the round brackets (cf lhs of (A.1), (A.2), etc, below). It will be

[6] We have assumed the relations $x_\mu x_\nu = x_\nu x_\mu$, $p_\mu p_\nu = p_\nu p_\mu$, $x_\mu p_\nu = q p_\nu x_\mu$, $\psi_\mu \psi_\nu + \psi_\nu \psi_\mu = 0$ in the phase space for the spinning relativistic particle where x_μ and p_μ are the target space (canonically conjugate) coordinates and momenta, respectively, and the fermionic Lorentz vector ψ_μ stands for the 'spin' degrees of freedom attached to the relativistic particle. In the above relations, the Lorentz invariance is respected for any arbitrary ordering of μ and ν. One of the highlights of this work is the $GL_{\sqrt{q}}(1|1)$ and $GL_q(2)$ invariance of the solutions for the equations of motion at any arbitrary value of the parameter (i.e. time) of the evolution. The equations of motion are derived from the Euler–Lagrange equations.

noted that, on the rhs, there is no such reordering because the trilinears on the rhs are *unique* in the sense that all the tilde oscillators have been brought to the left due to the algebras of section 3 for comparison. In fact, it is the comparison on the rhs of the reordering[7] that determines the exact values for the parameters λ and ν. To corroborate the above assertion, we take here a set of four trilinears of the (super)oscillators and show that the requirement of the associativity condition yields the same values for λ and ν for all members of this set. In fact, the rearrangements of the four members of the trilinears of the (super)oscillators,

$$\begin{aligned} A(\tilde{A}\tilde{B}) &= \left[\left(\frac{\lambda-\nu}{pq}\right)\right]\tilde{B} + \left[\left(\frac{\lambda(\lambda-\nu)}{pq}\right)\right]\tilde{B}\tilde{A}A \\ (A\tilde{A})\tilde{B} &= [(1+\nu)]\,\tilde{B} + \left[\frac{\lambda(\lambda-\nu)}{pq} + \nu\left(\lambda - \frac{\lambda-\nu}{pq}\right)\right]\tilde{B}\tilde{A}A \end{aligned} \tag{A.1}$$

$$\begin{aligned} B(\tilde{A}\tilde{B}) &= \left[\frac{1}{p}\left(1+\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A} + \left[\frac{\lambda}{p}\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{A}A \\ &\quad + \left[\frac{\lambda}{p}\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{B}B \\ (B\tilde{A})\tilde{B} &= \left[\left(\frac{\lambda-\nu}{p}\right)\right]\tilde{A} + \left[\left(\frac{\lambda-\nu}{p}\right)\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{A}A \\ &\quad + \left[\left(\frac{\lambda-\nu}{p}\right)\left(\nu-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}\tilde{B}B \end{aligned} \tag{A.2}$$

$$\begin{aligned} A(B\tilde{B}) &= \left[1+\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]A + \left[\lambda\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}AA \\ &\quad + \left[\nu\left(\lambda-\frac{\lambda-\nu}{pq}\right) + (\lambda-\nu)\left(\nu-\frac{\lambda-\nu}{pq}\right)\right]\tilde{B}BA \\ (AB)\tilde{B} &= [(\lambda-\nu)]\,A + \left[(\lambda-\nu)\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}AA + \left[(\lambda-\nu)\left(\nu-\frac{\lambda-\nu}{pq}\right)\right]\tilde{B}BA \end{aligned} \tag{A.3}$$

$$\begin{aligned} B(A\tilde{A}) &= [(1+\nu)]\,B + \left[\frac{\lambda(\lambda-\nu)}{pq} + \nu\left(\lambda-\frac{\lambda-\nu}{pq}\right)\right]\tilde{A}AB \\ (BA)\tilde{A} &= \left[\left(\frac{\lambda-\nu}{pq}\right)\right]B + \left[\left(\frac{\lambda(\lambda-\nu)}{pq}\right)\right]\tilde{A}AB \end{aligned} \tag{A.4}$$

lead to a unique set of relations between λ and ν

$$\lambda = pq + (pq+1)\nu \qquad \nu\left(\lambda - \frac{\lambda-\nu}{pq}\right) = 0 \tag{A.5}$$

when the rhs of the above equations (i.e. (A.1)–(A.4)) are matched with each other. It is straightforward to check that the above relations lead to the set of values of λ and ν as quoted in (4.3). This discussion demonstrates the uniqueness of the algebras (3.8), (4.4) and (4.5). Of course, the algebra (4.6) is a special case of the above algebras when $pq = 1$.

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